Chemomechanical Simulation of Soap Film Flow on Spherical Bubbles: Supplemental Document

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ABSTRACT

This document provides a detailed derivation of the kinematic condition and the lubrication model, i.e., Equation (10) in the main paper.

1 THE KINEMATIC CONDITION

In this section we derive the kinematic condition in spherical coordinates (Equation (7)), all the quantities preserve their physical dimensions. Let $F(\theta, \phi, r, t) = 0$ be an implicit definition of the outer bubble surface, where η is the half thickness of the film. Then, the height field at the outer side of the bubble can be written as $r = R + \eta(\theta, \phi, t)$ and the free surface is given by

$$F(\theta, \phi, r, t) = \eta(\theta, \phi, t) - r + R = 0. \tag{48}$$

Taking the material derivative of F we have

$$\begin{aligned} 0 &= \frac{DF}{Dt} \\ &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial F}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial F}{\partial r}\frac{dr}{dt} \\ &= \frac{\partial F}{\partial t} + \frac{u_\theta}{R}\frac{\partial F}{\partial \theta} + \frac{u_\phi}{R\sin\theta}\frac{\partial F}{\partial \phi} + u_r\frac{\partial F}{\partial r} \\ &= \frac{\partial(\eta - r)}{\partial t} + \frac{u_\theta}{R}\frac{\partial(\eta - r)}{\partial \theta} + \frac{u_\phi}{R\sin\theta}\frac{\partial(\eta - r)}{\partial \phi} + u_r\frac{\partial(\eta - r)}{\partial r} \\ &= \left(\frac{\partial \eta}{\partial t} - 0\right) + \frac{u_\theta}{R}\left(\frac{\partial \eta}{\partial \theta} - 0\right) + \frac{u_\phi}{R\sin\theta}\left(\frac{\partial \eta}{\partial \phi} - 0\right) + u_r\,(0 - 1) \\ &= \frac{\partial \eta}{\partial t} + \frac{u_\theta}{R}\frac{\partial \eta}{\partial \theta} + \frac{u_\phi}{R\sin\theta}\frac{\partial \eta}{\partial \phi} - u_r \end{aligned} \tag{49}$$

and subsequently

$$u_r = \frac{\partial \eta}{\partial t} + \frac{u_\theta}{R}\frac{\partial \eta}{\partial \theta} + \frac{u_\phi}{R\sin\theta}\frac{\partial \eta}{\partial \phi}. \tag{50}$$

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The derivation for the inner side with $r = R - \eta(\theta, \phi, t)$ is similar, yielding

$$-u_r = \frac{\partial \eta}{\partial t} + \frac{u_\theta}{R}\frac{\partial \eta}{\partial \theta} + \frac{u_\phi}{R\sin\theta}\frac{\partial \eta}{\partial \phi}. \tag{51}$$

2 ASYMPTOTIC EXPANSION

In this section, we derive Equation (10), with neglect of the force term $\mathbf{f}$.

The Cauchy stress tensor σ in Equation (1) is defined in terms of local fluid pressure and viscosity as $\sigma = -p\mathbf{I} + \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top]$; in spherical coordinates it has components

$$\sigma_{\theta\theta} = -p + \frac{2\mu}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \tag{52a}$$

$$\sigma_{\phi\phi} = -p + \frac{2\mu}{r\sin\theta}\left(\frac{\partial u_\phi}{\partial \phi} + u_\theta\cos\theta + u_r\sin\theta\right), \tag{52b}$$

$$\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r}, \tag{52c}$$

$$\sigma_{\theta\phi} = \frac{\mu}{r\sin\theta}\left(\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta}\sin\theta - u_\phi\cos\theta\right), \tag{52d}$$

$$\sigma_{\theta r} = \frac{\mu}{r}\left[r\frac{\partial u_\theta}{\partial r} - u_\theta + \frac{\partial u_r}{\partial \theta}\right], \tag{52e}$$

$$\sigma_{\phi r} = \frac{\mu}{r\sin\theta}\left[r\frac{\partial u_\phi}{\partial r}\sin\theta - u_\phi\sin\theta + \frac{\partial u_r}{\partial \phi}\right]. \tag{52f}$$

The stress boundary conditions at both surfaces are

$$\sigma \cdot \mathbf{n}_{i,o} = (2\mathscr{C}_{i,o}\gamma - p_{i,o})\mathbf{n}_{i,o} + \nabla_s\gamma, \tag{53}$$

where $p_{i,o}$ represent air pressure at the inner and the outer surface, and $2\mathscr{C}_{i,o} = -\nabla \cdot \mathbf{n}_{i,o}$ is twice the mean surface curvature, and the outward unit normal vectors to both film surfaces are

$$\mathbf{n}_o = \frac{-\frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial \eta}{\partial \phi}\mathbf{e}_\phi + \mathbf{e}_r}{\sqrt{1 + \frac{1}{r^2}\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \eta}{\partial \phi}\right)^2}}, \tag{54}$$

$$\mathbf{n}_i = \frac{-\frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_\theta - \frac{1}{r\sin\theta}\frac{\partial \eta}{\partial \phi}\mathbf{e}_\phi - \mathbf{e}_r}{\sqrt{1 + \frac{1}{r^2}\left(\frac{\partial \eta}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \eta}{\partial \phi}\right)^2}}, \tag{55}$$

where as the (not normalized) tangent vectors are

$$\mathbf{t}_{o1} = \mathbf{e}_\theta + \frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_r, \quad \mathbf{t}_{o2} = \mathbf{e}_\phi + \frac{1}{r\sin\theta}\frac{\partial \eta}{\partial \phi}\mathbf{e}_r, \tag{56}$$

$$\mathbf{t}_{i1} = \mathbf{e}_\theta - \frac{1}{r}\frac{\partial \eta}{\partial \theta}\mathbf{e}_r, \quad \mathbf{t}_{i2} = \mathbf{e}_\phi - \frac{1}{r\sin\theta}\frac{\partial \eta}{\partial \phi}\mathbf{e}_r. \tag{57}$$

With the variables in Equations (1), (5) and (52) non-dimensionalized by

$$\begin{aligned} \epsilon &= \frac{\eta_0}{R}, \quad \eta = \eta_0 \eta', \quad u_\theta = U u_\theta', \quad u_\phi = U u_\phi', \\ \sigma &= \frac{\mu U}{R}\sigma', \quad u_r = \epsilon U u_r', \quad r = R + \epsilon R r', \\ t &= \frac{R}{U}t', \quad \Gamma = \Gamma_0 \Gamma', \quad p = \frac{\mu U}{R\epsilon}p', \quad D_s = URD_s', \end{aligned} \tag{58}$$

and dropping the primes of the non-dimensionalized quantities for readability, we arrive at

$$\frac{\partial u_\theta}{\partial t} + u_\theta \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + u_r \frac{\partial u_\theta}{\partial r} + \epsilon u_\theta u_r - \frac{u_\phi^2}{\tan\theta} \tag{59a}$$
$$= Re^{-1}\left(\epsilon^{-1}\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{\partial \sigma_{\theta\theta}\sin\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi} + 3\sigma_{\theta r} - \frac{\sigma_{\phi\phi}}{\tan\theta}\right),$$

$$\frac{\partial u_\phi}{\partial t} + u_\theta \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{\sin\theta}\frac{\partial u_\phi}{\partial \phi} + u_r \frac{\partial u_\phi}{\partial r} + \epsilon u_\phi u_r + \frac{u_\theta u_\phi}{\tan\theta} \tag{59b}$$
$$= Re^{-1}\left(\epsilon^{-1}\frac{\partial \sigma_{\phi r}}{\partial r} + \frac{\partial \sigma_{\theta\phi}\sin\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + 3\sigma_{\phi r} + \frac{\sigma_{\theta\phi}}{\tan\theta}\right),$$

$$\epsilon\left(\frac{\partial u_r}{\partial t} + u_\theta \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{\sin\theta}\frac{\partial u_r}{\partial \phi} + u_r \frac{\partial u_r}{\partial r}\right) + u_\theta^2 + u_\phi^2 \tag{59c}$$
$$= Re^{-1}\left(\epsilon^{-1}\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{\theta r}\sin\theta}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} + 2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right),$$

$$\frac{\partial u_\theta \sin\theta}{\partial\theta} + \frac{\partial u_\phi}{\partial\phi} + \sin\theta\frac{\partial u_r}{\partial r} = 0, \tag{59d}$$

$$\frac{\partial\Gamma}{\partial t} + \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\Gamma u_\theta \sin\theta\right) + \frac{\partial}{\partial\phi}\left(\Gamma u_\phi\right)\right] \tag{59e}$$
$$= \frac{D_s}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Gamma}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Gamma}{\partial\phi}\right)\right].$$

Additionally, we introduce the following dimensionless numbers

$$S = \frac{\gamma_a \epsilon}{\mu U}, \quad Re = \frac{UR\rho}{\mu}, \quad M = \frac{\Gamma_0 \gamma_r}{\rho \eta_0 U^2}, \tag{60}$$

where S is a measure of the equilibrium surface tension, Re is the Reynolds number and M is the Marangoni number. The surface tension is now given by

$$\gamma = \mu U\left(\epsilon^{-1}S - MRe\Gamma\right), \tag{61}$$

and the expression for $2\mathscr{C}_o$ is

$$2\mathscr{C}_o = \frac{\epsilon}{R}\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\eta - \frac{2}{R} + \frac{2\epsilon\eta}{R} + O(\epsilon^2), \tag{62}$$

whereas that for $2\mathscr{C}_i$ is similar and not to be explicitly listed.

We perform an asymptotic analysis by expanding the variables in ϵ series; that is, we expand $\sigma_{\theta r}, \sigma_{\phi r}, \sigma_{rr}$ as

$$f = \epsilon^{-1} f^{(-1)} + f^{(0)} + \epsilon f^{(1)} + O(\epsilon^2), \tag{63}$$

and $\mathbf{u}, \Gamma, p, \eta$ as well as the other components of the stress tensor as

$$g = g^{(0)} + \epsilon g^{(1)} + O(\epsilon^2), \tag{64}$$

Substituting the above expansions into Equation (59), grouping terms with equal powers of ϵ and setting the coefficients to zero, we obtain a series of solvable linear equations.

2.1 Leading-order Equations

The leading-order problem consists of the continuity equation

$$\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta} + \frac{\partial u_\phi^{(0)}}{\partial\phi} + \sin\theta\frac{\partial u_r^{(0)}}{\partial r} = 0, \tag{65}$$

and the momentum equations

$$Re^{-1}\frac{\partial\sigma_{\theta r}^{(-1)}}{\partial r} = 0, \quad Re^{-1}\frac{\partial\sigma_{\phi r}^{(-1)}}{\partial r} = 0, \quad Re^{-1}\frac{\partial\sigma_{rr}^{(-1)}}{\partial r} = 0. \tag{66}$$

For the boundary conditions, we have the kinematic condition (see Section 1) at the outer boundary

$$u_r^{(0)} = \frac{\partial\eta^{(0)}}{\partial t} + u_\theta^{(0)}\frac{\partial\eta^{(0)}}{\partial\theta} + \frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial\eta^{(0)}}{\partial\phi} \tag{67}$$

and at the inner boundary

$$-u_r^{(0)} = \frac{\partial\eta^{(0)}}{\partial t} + u_\theta^{(0)}\frac{\partial\eta^{(0)}}{\partial\theta} + \frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial\eta^{(0)}}{\partial\phi}, \tag{68}$$

as well as the stress condition at the outer boundary

$$\sigma_{\theta r}^{(-1)} = 0, \quad \sigma_{\phi r}^{(-1)} = 0, \quad \sigma_{rr}^{(-1)} = -p_o - 2S, \tag{69}$$

and at the inner boundary

$$\sigma_{\theta r}^{(-1)} = 0, \quad \sigma_{\phi r}^{(-1)} = 0, \quad \sigma_{rr}^{(-1)} = -p_i + 2S. \tag{70}$$

Combining Equations (52), (66), (69) and (70), we discover that

$$\sigma_{\theta r}^{(-1)} = \frac{\partial u_\theta^{(0)}}{\partial r} = 0, \tag{71}$$

$$\sigma_{\phi r}^{(-1)} = \frac{\partial u_\phi^{(0)}}{\partial r} = 0, \tag{72}$$

$$\sigma_{rr}^{(-1)} = -p^{(0)} = -\frac{p_i + p_o}{2}. \tag{73}$$

Thus, $u_\theta^{(0)}$ and $u_\phi^{(0)}$ are independent of r and are only functions of θ, ϕ and t. As a result, we may integrate the continuity equation (65) in r and obtain

$$u_r^{(0)} = -\frac{r}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta} + \frac{\partial u_\phi^{(0)}}{\partial\phi}\right) + C(\theta, \phi, t), \tag{74}$$

combining with the kinematic conditions (67) and (68) yields

$$\frac{\partial\eta^{(0)}}{\partial t} + u_\theta^{(0)}\frac{\partial\eta^{(0)}}{\partial\theta} + \frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial\eta^{(0)}}{\partial\phi} + \frac{\eta^{(0)}}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta} + \frac{\partial u_\phi^{(0)}}{\partial\phi}\right) = 0. \tag{75}$$

The leading-order equation for the surfactant concentration is

$$\frac{\partial\Gamma^{(0)}}{\partial t}+\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\Gamma^{(0)}u_\theta^{(0)}\sin\theta)+\frac{\partial}{\partial\phi}(\Gamma^{(0)}u_\phi^{(0)})\right] \tag{76}$$
$$=\frac{D_s}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Gamma^{(0)}}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Gamma^{(0)}}{\partial\phi}\right)\right].$$

From the leading-order problem, we obtain two evolution equations (75) and (76) for the film thickness $\eta^{(0)}$ and the surfactant concentration $\Gamma^{(0)}$.

2.2 Remaining Orders

From the momentum equations and the stress boundary conditions we similarly obtain

$$\sigma_{\theta r}^{(0)}=\sigma_{\phi r}^{(0)}=0, \tag{77}$$
$$\sigma_{rr}^{(0)}=S\left(\frac{\partial^2}{\partial\theta^2}+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+2\right)\eta^{(0)} \tag{78}$$
$$=-p^{(1)}+2\frac{\partial u_r^{(0)}}{\partial r}.$$

Making use of Equations (52) and (78), we further obtain

$$\sigma_{\theta\theta}^{(0)}=-p^{(1)}+2\frac{\partial u_\theta^{(0)}}{\partial\theta} \tag{79}$$
$$=S\left(\frac{\partial^2}{\partial\theta^2}+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+2\right)\eta^{(0)}$$
$$+\frac{2}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta}+\frac{\partial u_\phi^{(0)}}{\partial\phi}\right)+2\frac{\partial u_\theta^{(0)}}{\partial\theta},$$

$$\sigma_{\phi\phi}^{(0)}=-p^{(1)}+2\left(\frac{1}{\sin\theta}\frac{\partial u_\phi^{(0)}}{\partial\phi}+\frac{u_\theta^{(0)}}{\tan\theta}\right) \tag{80}$$
$$=S\left(\frac{\partial^2}{\partial\theta^2}+\frac{1}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+2\right)\eta^{(0)}$$
$$+\frac{2}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta}+2\frac{\partial u_\phi^{(0)}}{\partial\phi}+u_\theta^{(0)}\cos\theta\right).$$

Now the leading-order momentum equations in Equation (59) can be rewritten as

$$\frac{\partial u_\theta^{(0)}}{\partial t}+u_\theta^{(0)}\frac{\partial u_\theta^{(0)}}{\partial\theta}+\frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial u_\theta^{(0)}}{\partial\phi}-\frac{(u_\phi^{(0)})^2}{\tan\theta} \tag{81a}$$
$$=Re^{-1}\left(\frac{\partial\sigma_{\theta r}^{(1)}}{\partial r}+\frac{\partial\sigma_{\theta\theta}^{(0)}\sin\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\sigma_{\theta\phi}^{(0)}}{\partial\phi}-\frac{\sigma_{\phi\phi}^{(0)}}{\tan\theta}\right),$$

$$\frac{\partial u_\phi^{(0)}}{\partial t}+u_\theta^{(0)}\frac{\partial u_\phi^{(0)}}{\partial\theta}+\frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial u_\phi^{(0)}}{\partial\phi}+\frac{u_\theta^{(0)}u_\phi^{(0)}}{\tan\theta} \tag{81b}$$
$$=Re^{-1}\left(\frac{\partial\sigma_{\phi r}^{(1)}}{\partial r}+\frac{\partial\sigma_{\theta\phi}^{(0)}\sin\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\sigma_{\phi\phi}^{(0)}}{\partial\phi}+\frac{\sigma_{\theta\phi}^{(0)}}{\tan\theta}\right).$$

The only unknowns remaining are $\sigma_{\theta r}^{(1)}$ and $\sigma_{\phi r}^{(1)}$. To obtain them, we write the two tangential stress conditions in $O(\epsilon)$ at the outer boundary as

$$\frac{\partial\eta}{\partial\theta}\left(\sigma_{rr}^{(0)}-\sigma_{\theta\theta}^{(0)}\right)-\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\phi}\sigma_{\theta\phi}^{(0)}+\sigma_{\theta r}^{(1)}=-MRe\frac{\partial\Gamma^{(0)}}{\partial\theta}, \tag{82}$$
$$\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\phi}\left(\sigma_{rr}^{(0)}-\sigma_{\phi\phi}^{(0)}\right)-\frac{\partial\eta}{\partial\theta}\sigma_{\theta\phi}^{(0)}+\sigma_{\phi r}^{(1)}=-\frac{MRe}{\sin\theta}\frac{\partial\Gamma^{(0)}}{\partial\phi}. \tag{83}$$

All the zero-order terms in Equation (81) do not vary with r. Integrating Equation (81) in r and applying Equations (82) and (83), then combining with Equations (52) and (65), we obtain the evolution equations for $u_\theta^{(0)}$ and $u_\phi^{(0)}$ to close our system

$$\frac{\partial u_\theta^{(0)}}{\partial t}+u_\theta^{(0)}\frac{\partial u_\theta^{(0)}}{\partial\theta}+\frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial u_\theta^{(0)}}{\partial\phi}-\frac{(u_\phi^{(0)})^2}{\tan\theta} \tag{84a}$$
$$=-\frac{M}{\eta^{(0)}}\frac{\partial\Gamma^{(0)}}{\partial\theta}$$
$$+Re^{-1}\left\{\frac{1}{\eta^{(0)}\sin\theta}\frac{\partial\eta^{(0)}}{\partial\phi}\sigma_{\theta\phi}^{(0)}\right.$$
$$+\frac{2}{\eta^{(0)}}\frac{\partial\eta^{(0)}}{\partial\theta}\left[\frac{\partial u_\theta^{(0)}}{\partial\theta}+\frac{1}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta}+\frac{\partial u_\phi^{(0)}}{\partial\phi}\right)\right]$$
$$\left.+\frac{\partial\sigma_{\theta\theta}^{(0)}\sin\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\sigma_{\theta\phi}^{(0)}}{\partial\phi}-\frac{\sigma_{\phi\phi}^{(0)}}{\tan\theta}\right\},$$

$$\frac{\partial u_\phi^{(0)}}{\partial t}+u_\theta^{(0)}\frac{\partial u_\phi^{(0)}}{\partial\theta}+\frac{u_\phi^{(0)}}{\sin\theta}\frac{\partial u_\phi^{(0)}}{\partial\phi}+\frac{u_\theta^{(0)}u_\phi^{(0)}}{\tan\theta} \tag{84b}$$
$$=-\frac{M}{\eta^{(0)}\sin\theta}\frac{\partial\Gamma^{(0)}}{\partial\phi}$$
$$+Re^{-1}\left\{\frac{1}{\eta^{(0)}}\frac{\partial\eta^{(0)}}{\partial\theta}\sigma_{\theta\phi}^{(0)}\right.$$
$$+\frac{2}{\eta^{(0)}\sin^2\theta}\frac{\partial\eta^{(0)}}{\partial\phi}\left[\frac{\partial u_\phi^{(0)}}{\partial\phi}+u_\theta^{(0)}\cos\theta+\frac{\partial u_\theta^{(0)}\sin\theta}{\partial\theta}+\frac{\partial u_\phi^{(0)}}{\partial\phi}\right]$$
$$\left.+\frac{\partial\sigma_{\theta\phi}^{(0)}\sin\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial\sigma_{\phi\phi}^{(0)}}{\partial\phi}+\frac{\sigma_{\theta\phi}^{(0)}}{\tan\theta}\right\},$$

where $\sigma_{\theta\theta}^{(0)}$ and $\sigma_{\phi\phi}^{(0)}$ are given by Equation (79), and

$$\sigma_{\theta\phi}^{(0)}=\frac{1}{\sin\theta}\left(\frac{\partial u_\theta^{(0)}}{\partial\phi}+\frac{\partial u_\phi^{(0)}}{\partial\theta}\sin\theta-u_\phi^{(0)}\cos\theta\right). \tag{85}$$

The full evolution equations are thus Equations (75), (76) and (84). If we denote terms in Equation (84) associated with Re^{-1} as $\mathbf{V}$, and drop the superscript $^{(0)}$, the governing equations can be simplified to

$$\begin{cases} \frac{D\mathbf{u}}{Dt} = -\frac{M}{\eta}\nabla\Gamma + Re^{-1}\mathbf{V}, & (86a) \\ \frac{D\Gamma}{Dt} = -\Gamma\nabla\cdot\mathbf{u} + D_s\nabla^2\Gamma, & (86b) \\ \frac{D\eta}{Dt} = -\eta\nabla\cdot\mathbf{u}, & (86c) \end{cases}$$

with $\mathbf{u} = (u_\theta, u_\phi)^\top$. This corresponds to Equation (10).